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Technical section

A controlled clothoid spline

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Abstract

A clothoid has the property that its curvature varies linearly with arclength. This is a useful feature for the path of a vehicle whose turning radius is controlled as a linear function of the distance travelled. Highways, railways and the paths of car-like robots may be composed of straight line segments, clothoid segments and circular arcs. Control polylines are used in computer aided design and computer aided geometric design applications to guide composite curves during the design phase. This article examines the use of a control polyline to guide a curve composed of segments of clothoids, straight lines, and circular arcs.

Keywords: Clothoid; Spline; Control polyline

1. Introduction

The curvature of a clothoid varies linearly with its arclength. This property makes it particularly suitable for the trajectory of a vehicle which has a controlling mechanism that turns its wheels based on a linear function of the distance travelled [1,2]. The suitability of clothoid-based trajectories for car-like robots is studied extensively in [3]. Control polylines have been used for decades in computer aided design (CAD) and computer aided geometric design (CAGD) applications to guide curves [4,5]. The advantage of a guided curve is the flexibility which it provides for interactive design and modification of curves using a graphics work station. Earlier work on guiding a clothoid spline was based on transition curves particular to highway design [6,7]. Further work constructed a planar curve of clothoid segments, circular arcs, and straight line segments such that it interpolated two given points while matching given tangent directions and curvatures at the points [8].

This paper examines a control polyline approach to guiding a clothoid spline. Arcs of clothoids and straight line segments which make up the spline are automatically determined and displayed. The radii of curvature at the turning points of the resulting curve are easily computed and displayed. The curve can be adjusted interactively should it be necessary to change the curvature at the turning points. Construction of the clothoid spline is based on using pairs of clothoids as blending curves. When a control point is moved, only three such pairs are affected. The control polyline approach has the advantage that it facilitates global curve design but allows for local fine-tuning.

A clothoid pair can be symmetric, i.e. the two clothoids are the same with one a reflection of the other, or unsymmetric, i.e. the two clothoids are different. A clothoid pair, symmetric or unsymmetric, can match any control polyline. Symmetric clothoid

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pairs will generally have straight line segments appended whereas unsymmetric clothoid pairs will generally have fewer (or no) straight line segments.

2. Notation and conventions

The usual Cartesian co-ordinate system with x- and y-axes is presumed. Positive angles are measured counter-clockwise. Boldface is used for points and vectors. The norm or length of a vector $\mathbf{A}$ is denoted as $||\mathbf{A}||$. The derivative of a function is denoted with a prime, e.g. $f'(t)$. To aid concise writing of mathematical expressions, the symbol $\times$ is used to denote the signed z-component of the usual 3D cross-product of two vectors in the x–y plane, i.e. $\mathbf{A} \times \mathbf{B} = A_x B_y - A_y B_x$. The tangent angle deviation is the angle through which the tangent vector rotates while one traverses the clothoid segment.

3. Background

3.1. Clothoid

A clothoid is defined in terms of the Fresnel integrals as [9]

$$x(\theta) = aC(\theta), \quad y(\theta) = aS(\theta),$$

where the scaling factor a is positive, the tangent angle deviation θ is non-negative, and the Fresnel integrals are [10]

$$C(\theta) = \frac{1}{\sqrt{2\pi}} \int_0^{\theta} \frac{\cos u}{\sqrt{u}}\, \mathrm{d}u, \quad S(\theta) = \frac{1}{\sqrt{2\pi}} \int_0^{\theta} \frac{\sin u}{\sqrt{u}}\, \mathrm{d}u.$$

The curvature of the above clothoid is

$$\kappa(\theta) = \sqrt{2\pi\theta}/a. \tag{1}$$

The following lemma is useful in proofs which will appear later in this article.

Lemma 1. *If* $0 < \theta < \pi$ *then*

$$\frac{C(\theta)}{S(\theta)} > \frac{\cos\theta}{\sin\theta}$$

and $C(\theta)/S(\theta)$ *is monotonically decreasing.*

Proof. Differentiation of $C(\theta)/S(\theta)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{C(\theta)}{S(\theta)}\right] = \frac{S(\theta)\cos\theta - C(\theta)\sin\theta}{\sqrt{2\pi\theta}[S(\theta)]^2}.$$

The numerator in the above may be expressed as

$$\frac{1}{\sqrt{2\pi}} \int_0^{\theta} \left[\frac{\sin u}{\sqrt{u}} \cos\theta - \frac{\cos u}{\sqrt{u}} \sin\theta\right] \mathrm{d}u = -\frac{1}{\sqrt{2\pi}} \int_0^{\theta} \frac{\sin(\theta - u)}{\sqrt{u}}\, \mathrm{d}u,$$

which is clearly less than zero for $0 < \theta < \pi$. □

3.2. Control polyline

The curve is derived from the control polyline using blending curves. Cubic B-spline blending curves are popular in traditional computer aided design applications because they allow inflection points and maintain curvature continuity at the joints. It is however difficult to control their curvature. Quadratic B-splines are simpler but do not maintain continuity of curvature. The clothoid spline derived from the control polyline uses pairs of clothoids as blending curves. It is similar to the traditional quadratic B-spline.

The blending curves of a quadratic B-spline are rotated and translated segments of parabolas. Each parabolic segment corresponds to an interior control polyline vertex, so if the vertices are labelled $\mathbf{P}_0, \ldots, \mathbf{P}_n$, then the parabolic segments are usually indexed from 1 to $n-1$. Each segment, except for the first and last, begins and ends at the midpoints of the edges adjacent to its corresponding vertex; the first segment begins at the first control vertex and the last segment ends at the last control vertex as illustrated in Fig. 1.

A control polyline for a clothoid spline is introduced by replacing each parabolic segment by a pair of clothoids joined at their point of highest curvature such that continuity of the unit tangent vector and curvature are preserved at the join. The clothoid is less flexible than a polynomial curve, so in some cases a straight line segment is appended to the clothoid pair along the longer adjacent edge of its corresponding control vertex. Thus it is always possible for a clothoid pair and a straight line segment to extend between the same two points as a parabolic segment. The straight line segment

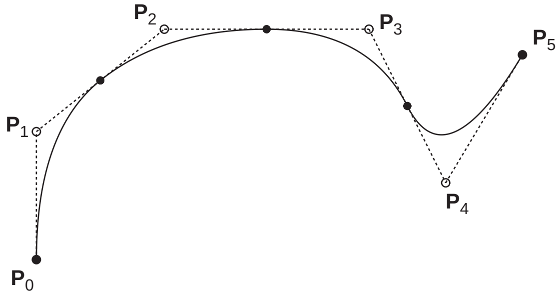

Fig. 1. A guided quadratic B-spline.

joins the clothoid in a tangent continuous manner at the point of zero curvature of the clothoid to give a G^2 join.

Each clothoid of the pair can be expressed in terms of its beginning point, $\mathbf{P}_0$, beginning unit tangent vector $\mathbf{T}_0$, and beginning unit normal vector $\mathbf{N}_0$ as $\mathbf{P}_0 + aC(\theta)\mathbf{T}_0 + aS(\theta)\mathbf{N}_0$, where a is a scaling factor, $0<\theta<\alpha<\pi$, α is the angle from $\mathbf{T}_0$ to the unit tangent vector at the joint of the pair, and $C(\theta)$ and $S(\theta)$ are the Fresnel integrals.

4. Blending with a clothoid pair

Two types of blending are considered, symmetric and unsymmetric. For symmetric blending the two clothoids in a blending pair are the same; one is simply a reflection of the other. For this type of blending it is necessary to append a straight line segment whenever the two adjacent edges of the corresponding vertex are unequal in length. For unsymmetric blending the two clothoids in a blending pair are different. In this case it is necessary to append a straight line segment only when there is a significant difference in length of the two adjacent edges of the corresponding vertex.

4.1. Symmetric blending

Given three distinct non-collinear points $\mathbf{P}_0$, $\mathbf{V}$ and $\mathbf{P}_1$ in the plane, where $||\mathbf{V}-\mathbf{P}_0|| = ||\mathbf{P}_1-\mathbf{V}|| = g$. Let $0<\alpha<\pi$ be the angle from $\mathbf{V}-\mathbf{P}_0$ to $\mathbf{P}_1-\mathbf{V}$ as shown in Fig. 2. It is desirable to have a smooth curvature continuous blending curve from $\mathbf{V}-\mathbf{P}_0$ to $\mathbf{P}_1-\mathbf{V}$. Such a blend has previously been considered.

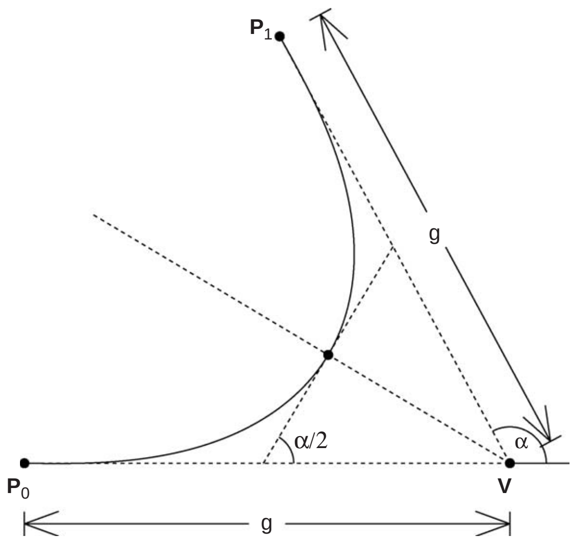

Fig. 2. Symmetric clothoid blending.

In [2] a point on the bisector of the angle formed by the adjacent edges of a vertex is chosen. A clothoid, whose beginning point (with zero curvature) is somewhere on one of the adjacent edges which meets the bisector such that its tangent vector is perpendicular to the bisector, is then computed. The second clothoid of the pair is determined by reflecting the first about the bisector.

The approach in this article is similar to that in [9]. The beginning point of the first clothoid is taken as $\mathbf{P}_0$ and its beginning direction is taken to be the same as the direction of $\mathbf{V}-\mathbf{P}_0$. To ensure tangential continuity, the tangent angle deviation at its endpoint is taken to be $\frac{1}{2}\alpha$. To ensure positional continuity the ending point is constrained to lie on the bisector of the angle formed by the adjacent edges of the corresponding vertex. This produces the equation

$$aC(\tfrac{1}{2}\alpha) + aS(\tfrac{1}{2}\alpha)\tan\tfrac{1}{2}\alpha = g$$

from which the scaling factor a can be solved; the scaling factor determines the clothoid. The second clothoid is obtained by reflecting the first about the bisector as illustrated in Fig. 2. Tangent vector and curvature continuity follow by symmetry. The maximum curvature is given by $\kappa(\alpha/2)$ in (1).

4.2. Unsymmetric blending

The following theorem gives conditions for non-symmetric blending.

Theorem 1. *Given three distinct non-collinear points* $\mathbf{P}_0$, $\mathbf{V}$ *and* $\mathbf{P}_1$ *in the plane. Let* $g = ||\mathbf{V}-\mathbf{P}_0||, \mathbf{T}_0 = (\mathbf{V}-\mathbf{P}_0)/g$, $h = ||\mathbf{V}-\mathbf{P}_1||, \mathbf{T}_1 = (\mathbf{V}-\mathbf{P}_1)/h$ *and let* $0<\alpha<\pi$ *be the angle from* $\mathbf{V}-\mathbf{P}_0$ *to* $\mathbf{P}_1-\mathbf{V}$. *Assume without loss of generality that* $\mathbf{P}_0$ *and* $\mathbf{P}_1$ *are labelled such that* $g>h$. *Define unit vectors* $\mathbf{N}_i$ *normal to* $\mathbf{T}_i$ *at* $\mathbf{P}_i$ *such that* $\mathbf{T}_0\times\mathbf{N}_0$ *and* $\mathbf{N}_1\times\mathbf{T}_1$ *are of the same sign as* $\mathbf{T}_1\times\mathbf{T}_0$. *Define* $k = g/h>1$. *If*

$$\frac{k+\cos\alpha}{\sin\alpha} < \frac{C(\alpha)}{S(\alpha)} \tag{2}$$

then a pair of clothoids A_i, $i = 0, 1$ *can be found which satisfy the following*:

1. A_i*'s beginning point* (*at which it has zero curvature*) *is at* $\mathbf{P}_i$,
2. A_i*'s unit normal vector at* $\mathbf{P}_i$ *coincides with* $\mathbf{N}_i$,
3. A_i*'s unit tangent vector at* $\mathbf{P}_i$ *coincides with* $\mathbf{T}_i$,
4. A_0 *and* A_1 *meet at point* $\mathbf{P}$, *and*
5. *at* $\mathbf{P}$ *they have the same unsigned curvature and their unit tangent vectors are parallel but in opposite directions*, *as shown in Fig.* 3.

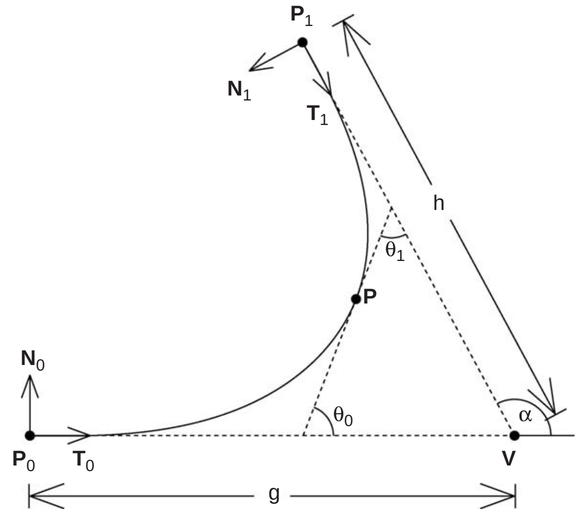

Fig. 3. Unsymmetric clothoid blending.

Proof. Note that (2) implies

$$\frac{1+k\cos\alpha}{k\sin\alpha}<\frac{C(\alpha)}{S(\alpha)} \tag{3}$$

since $(1+k\cos\alpha)/(k\sin\alpha)<(k+\cos\alpha)/(\sin\alpha)$. Choose the positions and orientations of A_i, $i=0,1$ such that items (1–3) of the theorem statement are satisfied. Let the scaling factor and tangent angle deviation of A_0 be a_0 and θ_0, respectively, and let those of A_1 be a_1 and θ_1, respectively. The tangent vectors of the two clothoids are parallel for $\theta_0+\theta_1=\alpha$. Using (1) it follows that they will have the same curvature when their tangent vectors are parallel for

$$a_1=a_0\sqrt{\frac{\theta_1}{\theta_0}}=a_0\sqrt{\frac{\alpha-\theta_0}{\theta_0}}. \tag{4}$$

It remains to establish that the condition for positional continuity at **P** can be satisfied. This condition may be expressed as

$$\begin{aligned}\mathbf{P}&=\mathbf{P}_0+a_0C(\theta_0)\mathbf{T}_0+a_0S(\theta_0)\mathbf{N}_0\\&=\mathbf{P}_1+a_1C(\theta_1)\mathbf{T}_1+a_1S(\theta_1)\mathbf{N}_1.\end{aligned}$$

Re-arranging, taking the dot product with $\mathbf{T}_0$ and $\mathbf{N}_0$, and using (4) to substitute a_1 in terms of a_0 yields

$$\begin{aligned}&a_0C(\theta_0)+a_0\sqrt{\frac{\alpha-\theta_0}{\theta_0}}C(\alpha-\theta_0)\cos\alpha\\&\quad+a_0\sqrt{\frac{\alpha-\theta_0}{\theta_0}}S(\alpha-\theta_0)\sin\alpha=g+h\cos\alpha\end{aligned}$$

and

$$\begin{aligned}&a_0S(\theta_0)+a_0\sqrt{\frac{\alpha-\theta_0}{\theta_0}}C(\alpha-\theta_0)\sin\alpha\\&\quad-a_0\sqrt{\frac{\alpha-\theta_0}{\theta_0}}S(\alpha-\theta_0)\cos\alpha=h\sin\alpha.\end{aligned}$$

Elimination of a_0 followed by some simplification gives the condition for positional continuity as that value of θ which satisfies

$$f(\theta_0)=0,\quad 0<\theta_0<\alpha<\pi, \tag{5}$$

where

$$\begin{aligned}f(\theta)&=\sqrt{\theta}[C(\theta)\sin\alpha-S(\theta)(k+\cos\alpha)]\\&\quad+\sqrt{\alpha-\theta}[S(\alpha-\theta)(1+k\cos\alpha)\\&\quad-kC(\alpha-\theta)\sin\alpha].\end{aligned} \tag{6}$$

Now

$$\begin{aligned}f(\tfrac{1}{2}\alpha)&=(1-k)\sqrt{\tfrac{1}{2}\alpha}\,[C(\tfrac{1}{2}\alpha)\sin\alpha\\&\quad+(1-\cos\alpha)S(\tfrac{1}{2}\alpha)]<0.\end{aligned}$$

Furthermore

$$\begin{aligned}f(\alpha)&=\sqrt{\alpha}[C(\alpha)\sin\alpha-S(\alpha)(k+\cos\alpha)]\\&=\sqrt{\alpha}S(\alpha)\sin\alpha\left[\frac{C(\alpha)}{S(\alpha)}-\frac{k+\cos\alpha}{\sin\alpha}\right],\end{aligned}$$

so, using (2)

$$f(\alpha)>0.$$

It thus follows that $f(\theta)$ has at least one zero. It follows from (6) by differentiation and subsequent simplification that

$$\begin{aligned}f'(\theta)&=\frac{[C(\theta)\sin\alpha-S(\theta)(k+\cos\alpha)]}{2\sqrt{\theta}}\\&\quad+\frac{[kC(\alpha-\theta)\sin\alpha-S(\alpha-\theta)(1+k\cos\alpha)]}{2\sqrt{\alpha-\theta}}\\&=\frac{S(\theta)\sin\alpha}{2\sqrt{\theta}}\left[\frac{C(\theta)}{S(\theta)}-\frac{k+\cos\alpha}{\sin\alpha}\right]\\&\quad+\frac{kS(\alpha-\theta)\sin\alpha}{2\sqrt{\alpha-\theta}}\left[\frac{C(\alpha-\theta)}{S(\alpha-\theta)}-\frac{1+k\cos\alpha}{k\sin\alpha}\right]\end{aligned}$$

or, by Lemma 1

$$\begin{aligned}f'(\theta)&>\frac{S(\theta)\sin\alpha}{2\sqrt{\theta}}\left[\frac{C(\alpha)}{S(\alpha)}-\frac{k+\cos\alpha}{\sin\alpha}\right]\\&\quad+\frac{kS(\alpha-\theta)\sin\alpha}{2\sqrt{\alpha-\theta}}\left[\frac{C(\alpha)}{S(\alpha)}-\frac{1+k\cos\alpha}{k\sin\alpha}\right].\end{aligned}$$

Hence, by (2) and (3), $f'(\theta)>0$ and $f(\theta)$ has a unique zero in $(0,\alpha)$. It can be found by a numerical technique e.g., bisection or Newton's method. □

Corollary 1.

$0<\theta_1<\frac{1}{2}\alpha<\theta_0$.

Proof. It follows immediately since $\theta_0>\frac{1}{2}\alpha$ and $\theta_0+\theta_1=\alpha$. $\square$

At the limit of inequality (2), i.e. when the "$<$" is replaced by "$=$", $\theta_0=\alpha$ is a solution to (5) in which case $\theta_1=0$ and $a_1=0$ so the shorter clothoid is of zero length. In practice, a small value for θ_1 could cause an undesirable large change in the radius of curvature over the length of clothoid A_1. To avoid this situation, the length of A_0 is restricted to be less than

$$g_{\lim}=h\left[\frac{C(\alpha)}{S(\alpha)}\sin\alpha-\cos\alpha\right],$$

but greater than h by appending a straight line segment to its beginning prior to fitting the clothoid pair. This is equivalent to moving $\mathbf{P}_0$ to

$$\tilde{\mathbf{P}}_0=\mathbf{V}-[(1-\tau)h+\tau g_{\lim}]\mathbf{T}_0,\quad 0\leqslant\tau<1.$$

Note that appending the straight line segment is only done when $g>g_{\mathrm{limm}}$. In practice a value of τ between 0.5 and 0.75 seems to work well. A value of 0.75 was used in all the examples presented here with the exception of Example 4 for which a value of 0.5 was used. Note that $\tau=0$ results in symmetric blending.

5. Circular arc insertion

An initial control polyline is obtained by placing at least three points on a computer screen using a pointing device such as a mouse. The blending clothoid pairs are then defined by the first and last points placed, and the midpoints of line segments joining consecutively placed points, as described earlier. When a control point is moved, the midpoints of up to two of its neighbouring edges are affected, so up to three clothoid pairs are affected. The minimum radius of curvature of the pair corresponding to the relocated control point will normally increase if the control polyline becomes flatter at the relocated point, otherwise it will decrease. Each of the two neighbouring pairs of clothoids exhibit similar behaviour. It may be desirable to change the minimum radius of curvature of only one clothoid pair while leaving the rest of the curve intact. This can be accomplished by the insertion of a circular arc.

Insertion of circular arcs to adjust the minimum radius of curvature locally are considered for both symmetric and unsymmetric blending.

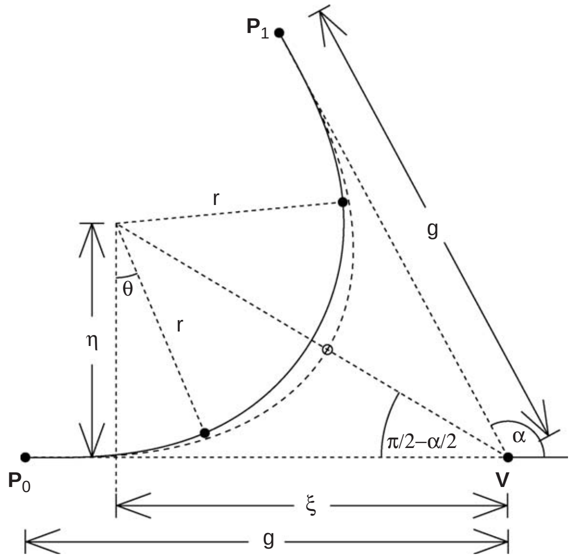

Fig. 4. Symmetric circular arc insertion.

5.1. Symmetric circular arc insertion

For this type of blending a circular arc is inserted between two symmetric clothoids. This is discussed in [11] for the case when the clothoids do not start at their points of zero curvature. Here, for the control polyline approach, the case when the clothoids have zero curvature at their starting points is considered. Only one clothoid is determined; the other is obtained by symmetry.

Given three non-collinear distinct points $\mathbf{P}_0$, $\mathbf{V}$ and $\mathbf{P}_1$ in the plane where $||\mathbf{V}-\mathbf{P}_0||=||\mathbf{V}-\mathbf{P}_1||=g$. Let α be the angle from $\mathbf{V}-\mathbf{P}_0$ to $\mathbf{P}_1-\mathbf{V}$ as for the symmetric blending case. Let the centre of the arc to be inserted be at $\mathbf{V}-\xi\mathbf{T}_0+\eta\mathbf{N}_0$ which by symmetry must lie on the bisector of the angle formed by the adjacent edges of $\mathbf{V}$ so $\eta/\xi=\cot\frac{1}{2}\alpha$ as shown in Fig. 4. Let the radius of the arc be r and let it meet the clothoid and have a common tangent where the clothoid's tangent angle deviation is θ. To ensure continuity of curvature at this point it follows from (1) that

$$a=r\sqrt{2\pi\theta}. \tag{7}$$

To ensure positional continuity it is required that

$$aS(\theta)+r\cos\theta=\eta$$

and

$$g-aC(\theta)+r\sin\theta=\xi.$$

Substitution of a in terms of r followed by elimination of η/ξ and some rearrangement yields

$$r=\frac{g}{\lambda(\theta)}, \tag{8}$$

where

$$\lambda(\theta) = \sqrt{2\pi\theta}[S(\theta)\tan\tfrac{1}{2}\alpha + C(\theta)] + \cos\theta\,\tan\tfrac{1}{2}\alpha - \sin\theta.$$

So, given a clothoid tangent angle deviation $0<\theta\leqslant\frac{1}{2}\alpha$, r is determined by (8) and a by (7). Now

$$\lambda'(\theta) = \sqrt{\frac{\pi}{2\theta}}[S(\theta)\tan\tfrac{1}{2}\alpha + C(\theta)]>0,$$

hence r is a monotone decreasing function of θ. The maximum value of r occurs when $\theta=0$ and is $r_{\max} = g\cot\frac{1}{2}\alpha$. The minimum value of r occurs when $\theta=\frac{1}{2}\alpha$, i.e. when no arc is inserted; it is $r_{\min} = g/\lambda(\frac{1}{2}\alpha)$. For r given within the above range, a corresponding value for θ can be obtained by solving the non-linear equation

$$F(\theta) = r\lambda(\theta) - g = 0.$$

This equation has a unique solution for $0<\theta<\frac{1}{2}\alpha$ since

$$F(0) = r\tan\tfrac{1}{2}\alpha - g < r_{\max}\tan\tfrac{1}{2}\alpha - g = 0,$$
$$F(\tfrac{1}{2}\alpha) = r\lambda(\tfrac{1}{2}\alpha) - g > r_{\min}\lambda(\tfrac{1}{2}\alpha) - g = 0$$

and F is monotone increasing. Once θ is known, r is determined by (8) and a by (7), from which the clothoid and circular arc to be inserted can be found.

5.2. Unsymmetric circular arc insertion

Specifying the radius of a circular arc to be inserted for this type of blending results in a system of two non-linear equations in two unknowns which is not easily solved. As an alternative, a tangent angle deviation for one of the clothoids may be specified.

Let $\theta_{0,u}$ and $\theta_{1,u}$ be the respective values obtained for θ_0 and θ_1 by unsymmetric blending under the hypothesis of Theorem 1. So $\theta_{0,u}+\theta_{1,u}=\alpha$. By analogy to the case of symmetric circular arc insertion, it is expected that the smallest radius of an inserted circle corresponds to an ending tangent angle deviation for A_1 which is $\theta_{1,u}$ otherwise a circular arc cannot be found such that the resulting curve is acceptable, e.g. the two clothoids may intersect as shown in Fig. 5. Also, by the same analogy, the maximum radius of an inserted circular arc corresponds to a value of $\theta_1=0$; this actually causes A_1 to be of zero length. These minimum and maximum values will be confirmed in the proof of Theorem 2 below. It is assumed that a curve designer will adjust the maximum radius of a clothoid pair by selecting a value of θ_1 between zero and $\theta_{1,u}$. A corresponding value for θ_0, $\theta_1<\theta_0<\alpha-\theta_1<\alpha-\theta_{1,u}=\theta_{0,u}$ is then determined by the following theorem:

Theorem 2. *Given three distinct non-collinear points* $\mathbf{P}_0$, $\mathbf{V}$ *and* $\mathbf{P}_1$ *in the plane, which satisfy the hypothesis of Theorem* 1. *A pair of clothoids* A_i, $i=0,1$, *and a circular arc* Ω *with endpoints* $\mathbf{Q}_i$, $i=0,1$ *can be found which satisfy the following*:

1. A_i, $i=0,1$, *satisfy items* (1)–(3) *of Theorem* 1,
2. A_i *meets and is tangent to* Ω *at the point* $\mathbf{Q}_i$, *and*
3. *at* $\mathbf{Q}_i$, A_i *has a radius of curvature equal to the radius of* Ω, *as shown in Fig.* 6.

Proof. Choose the positions and orientations of A_i, $i=0,1$ such that item (1) of the theorem statement is satisfied. Let the scaling factor and tangent angle of A_0 at $\mathbf{Q}_0$ be a_0 and θ_0, respectively, and let those of A_1 at $\mathbf{Q}_1$ be a_1 and θ_1, respectively. Let $\mathbf{N}_{Q,0}$ and $\mathbf{N}_{Q,1}$ be the unit normal vectors of A_0 and A_1 at $\mathbf{Q}_0$ and $\mathbf{Q}_1$, respectively. The centres of curvature of A_i at $\mathbf{Q}_i$, $i=0,1$ are

$$\mathbf{Z}_0 = \mathbf{P}_0 + a_0C(\theta_0)\mathbf{T}_0 + a_0S(\theta_0)\mathbf{N}_0 + \mathbf{N}_{Q,0}/\kappa(\theta_0) \quad (9)$$

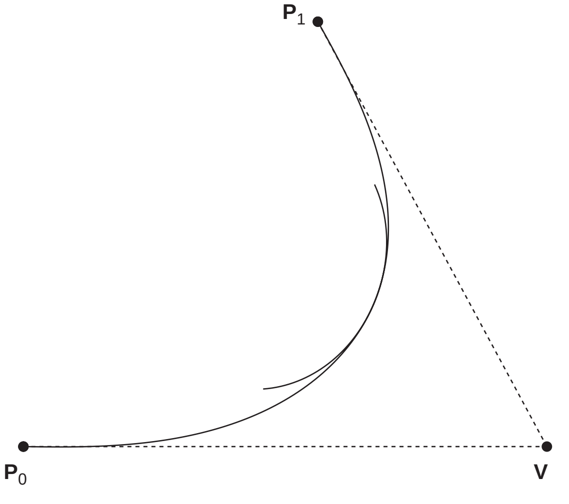

Fig. 5. Intersecting clothoids.

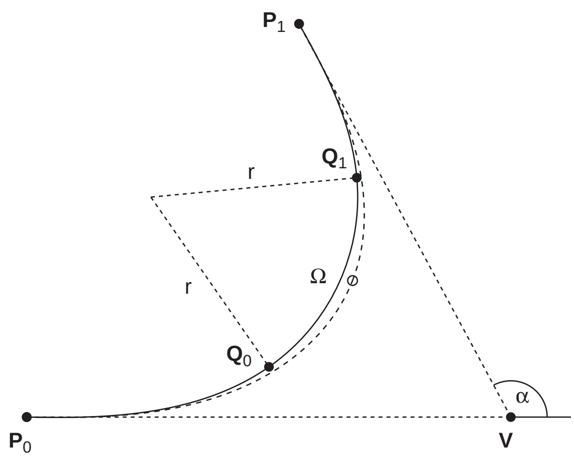

Fig. 6. Unsymmetric circular arc insertion.

and

$$\mathbf{Z}_1 = \mathbf{P}_1 + a_1 C(\theta_1)\mathbf{T}_1 + a_1 S(\theta_1)\mathbf{N}_1 + \mathbf{N}_{Q,1}/\kappa(\theta_1), \tag{10}$$

respectively. Items (3) and (2) are respectively satisfied for $\kappa(\theta_0) = \kappa(\theta_1)$ and $\mathbf{Z}_0 = \mathbf{Z}_1$ which, from (1), (9) and (10) may be written as

$$a_1 = a_0\sqrt{\frac{\theta_1}{\theta_0}}$$

and

$$a_0 C(\theta_0)\mathbf{T}_0 + a_0 S(\theta_0)\mathbf{N}_0 + \frac{a_0\mathbf{N}_{Q,0}}{\sqrt{2\pi\theta_0}} - a_0[C(\theta_1)\mathbf{T}_1 + S(\theta_1)\mathbf{N}_1]\sqrt{\frac{\theta_1}{\theta_0}} - \frac{a_0\mathbf{N}_{Q,1}}{\sqrt{2\pi\theta_0}} = \mathbf{P}_1 - \mathbf{P}_0. \tag{11}$$

Taking the dot product of (11) with $\mathbf{T}_0$ and $\mathbf{N}_0$ produces

$$a_0 C(\theta_0) - \frac{a_0 \sin\theta_0}{\sqrt{2\pi\theta_0}} + a_0[C(\theta_1)\cos\alpha + S(\theta_1)\sin\alpha] \times \sqrt{\frac{\theta_1}{\theta_0}} + \frac{a_0\sin(\alpha-\theta_1)}{\sqrt{2\pi\theta_0}} = g + h\cos\alpha \tag{12}$$

and

$$a_0 S(\theta_0) + \frac{a_0\cos\theta_0}{\sqrt{2\pi\theta_0}} + a_0[C(\theta_1)\sin\alpha - S(\theta_1)\cos\alpha] \times \sqrt{\frac{\theta_1}{\theta_0}} - \frac{a_0\cos(\alpha-\theta_1)}{\sqrt{2\pi\theta_0}} = h\sin\alpha. \tag{13}$$

Elimination of a_0 from (12) and (13), followed by rearrangement and some algebraic and trigonometric manipulation, shows that for a given θ_1, θ_0 satisfies the single non-linear equation in θ,

$$q(\theta) = 0, \tag{14}$$

where

$$\begin{aligned} q(\theta) = {} & \sqrt{2\pi\theta}[C(\theta)\sin\alpha - S(\theta)\cos\alpha - kS(\theta)] \\ & - \sqrt{2\pi\theta_1}[kC(\theta_1)\sin\alpha - kS(\theta_1)\cos\alpha - S(\theta_1)] \\ & - \cos(\alpha-\theta) - k\cos\theta \\ & + \cos\theta_1 + k\cos(\alpha-\theta_1). \end{aligned} \tag{15}$$

Now

$$\begin{aligned} q(\theta_1) = (1-k)\Big\{ & \sqrt{2\pi\theta_1}[C(\theta_1)\sin\alpha - S(\theta_1) \\ & \times\cos\alpha - S(\theta_1)] + 2\sin\tfrac{1}{2}\alpha\sin(\tfrac{1}{2}\alpha-\theta_1)\Big\}. \end{aligned}$$

From Lemma 1 and Theorem 1, Corollary 1

$$C(\theta_1)\sin\alpha - S(\theta_1)\cos\alpha - S(\theta_1) > 2\frac{S(\theta_1)}{\sin\theta_1}\cos\tfrac{1}{2}\alpha\sin(\tfrac{1}{2}\alpha-\theta_1) > 0$$

and $\sin\frac{1}{2}\alpha\,\sin(\frac{1}{2}\alpha-\theta_1) > 0$, so $q(\theta_1) < 0$. Furthermore

$$\begin{aligned} q(\alpha-\theta_1) = {} & \sqrt{2\pi(\alpha-\theta_1)}[C(\alpha-\theta_1)\sin\alpha \\ & - S(\alpha-\theta_1)\cos\alpha - kS(\alpha-\theta_1)] \\ & - \sqrt{2\pi\theta_1}[kC(\theta_1)\sin\alpha \\ & - kS(\theta_1)\cos\alpha - S(\theta_1)]. \end{aligned}$$

By comparison with (6), $q(\alpha-\theta_1)$ is the same as $\sqrt{2\pi}f(\alpha-\theta_1)$, which is a monotone increasing function of $\alpha-\theta_1$ with $f(\alpha-\theta_{1,u}) = f(\theta_{0,u}) = 0$. So $q(\alpha-\theta_1) > 0$ since $\theta_1 < \theta_{1,u}$.

Finally, for a given (fixed) θ_1,

$$q'(\theta) = \sqrt{\frac{\pi}{2\theta}}S(\theta)\sin\alpha\left[\frac{C(\theta)}{S(\theta)} - \frac{k+\cos\alpha}{\sin\alpha}\right],$$

so if (2) holds then $q'(\theta) > 0$. There is thus a unique value of θ which satisfies (14). It can be found by a numerical technique e.g. bisection or Newton's method. □

To get an idea of how r varies with relation to θ_1, it is helpful to look at the relationship between θ_1 and θ. The latter relationship is defined by (14) and (15). Implicit differentiation gives

$$\frac{d\theta}{d\theta_1} = k\frac{S(\theta_1)}{S(\theta)}\frac{\dfrac{C(\theta_1)}{S(\theta_1)} - \dfrac{1+k\cos\alpha}{k\sin\alpha}}{\dfrac{C(\theta)}{S(\theta)} - \dfrac{k+\cos\alpha}{\sin\alpha}}\sqrt{\frac{\theta}{\theta_1}}.$$

If (2) is satisfied, then from (3) and Lemma 1 since θ, $\theta_1 < \alpha$

$$\frac{C(\theta_1)}{S(\theta_1)} > \frac{C(\alpha)}{S(\alpha)} > \frac{1+k\cos\alpha}{k\sin\alpha}$$

and

$$\frac{C(\theta)}{S(\theta)} > \frac{C(\alpha)}{S(\alpha)} > \frac{k+\cos\alpha}{\sin\alpha}.$$

Hence $d\theta/d\theta_1 > 0$.

For the radius r of Ω, when $\theta = \theta_0$ it follows from (7) and (13) that

$$r = \frac{h\sin\alpha}{\psi(\theta_0,\theta_1)},$$

where

$$\begin{aligned} \psi(\theta_0,\theta_1) = {} & \sqrt{2\pi\theta_0}S(\theta_0) + \cos\theta_0 \\ & + \sqrt{2\pi\theta_1}[C(\theta_1)\sin\alpha - S(\theta_1)\cos\alpha] \\ & - \cos(\alpha-\theta_1). \end{aligned}$$

Now

$$\frac{dr}{d\theta_1} = -\frac{h\sin\alpha}{[\psi(\theta_0,\theta_1)]^2}\frac{d}{d\theta_1}\psi(\theta_0,\theta_1),$$

where

$$\frac{d}{d\theta_1}\psi(\theta_0,\theta_1) = \sqrt{\frac{\pi}{2\theta_0}}S(\theta_0)\frac{d\theta_0}{d\theta_1} + \sqrt{\frac{\pi}{2\theta_1}}S(\theta_1)\sin\alpha\left[\frac{C(\theta_1)}{S(\theta_1)} - \frac{\cos\alpha}{\sin\alpha}\right].$$

For the fourth factor (in square brackets) of the second term it follows by Lemma 1 (since $\theta_1 < \alpha$) that

$$\frac{C(\theta_1)}{S(\theta_1)} > \frac{C(\alpha)}{S(\alpha)} > \frac{\cos\alpha}{\sin\alpha}.$$

It thus follows that $d\psi(\theta_0,\theta_1)/d\theta_1 > 0$ and hence $dr/d\theta_1 < 0$, i.e. r is a monotonically decreasing function of θ_1. So the radius of the inserted circle can be increased by decreasing the value of θ_1.

6. Examples

Practical applications which use clothoids include highway design and robot path planning. In highway design the route is constrained by the terrain or environmental considerations and is also subject to further geometric constraints, in particular, the requirement to abide by given minimum radii of curvature [12]. In robot path planning the constraints are the locations of obstacles to be avoided, and minimum turning radii of the robots [13]. The advantage of using a control polyline for such applications is that, in an interactive graphics environment, after an initial highway route or robot path has been defined, it can be dynamically modified by adjusting the vertices of the control polyline. The designer is then able to immediately visualize the changes to the route or robot path. Furthermore, the minimum radii at critical points can also be displayed for the designer to immediately see the effect of the changes in the route or path.

Examples 1–3 demonstrate some basic operations which can be performed by using the proposed control polyline approach. Example 4 demonstrates the proposed approach as an alternative to that used in [6] for highway design. Example 5 shows how the control polyline can be used to design a path that avoids obstacles.

Example 1. This is a simple example to illustrate placement of control vertices and the resulting curve as shown in Fig. 7. Unsymmetric blending is used. There are four interior control vertices and four corresponding clothoid pairs. It can be noted that the only straight line segment occurs at the end, joining the last clothoid pair to the last control vertex. Had symmetric blending been used, then a straight line segment would occur for each clothoid pair since the edges of the control polyline are of unequal lengths. The coordinates of the vertices are (using dimensionless units):

$$\mathbf{P}_0 = (1.75,\ 2.75),\quad \mathbf{P}_1 = (1.75,\ 4),\quad \mathbf{P}_2 = (3,\ 5),$$
$$\mathbf{P}_3 = (5,\ 5),\quad \mathbf{P}_4 = (5.75,\ 3.5),\quad \mathbf{P}_5 = (6.5,\ 4.75).$$

The minimum radii of curvature of the clothoid pairs corresponding to the control vertices $\mathbf{P}_1, \ldots, \mathbf{P}_4$ are: 1.085, 1.297, 0.766 and 0.312.

Example 2. For this example vertex $\mathbf{P}_3$ of Example 1 was moved to position (4.5, 4.75) as shown in Fig. 8. This affected the three clothoid pairs corresponding to vertices $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$; their minimum radii changed to 0.889, 1.294 and 0.468.

Example 3. For this example a circular arc was inserted between the two clothoids of the pair corresponding to vertex $\mathbf{P}_4$ as shown in Fig. 9. The allowable tangent angle deviation range of the shorter clothoid was from 0° to 10.9°. A value of 5.7° was chosen causing the minimum radius to change to 0.55 from 0.468.

Example 4. For this example, a part of the designed highway alignment example in [6], was extracted. The control polyline is shown with dashed lines in Fig. 10. In the original example, circles with radii 40.9 and 58.4 units were specified corresponding to vertices $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively. Two clothoid segments were used to form a smooth C-shaped curvature-continuous transition

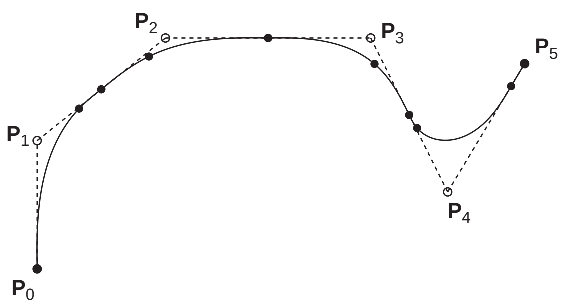

Fig. 7. Example 1—Initial control polyline.

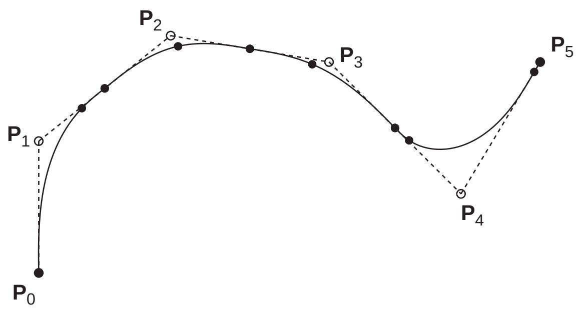

Fig. 8. Example 2—Vertex $\mathbf{P}_3$ repositioned.

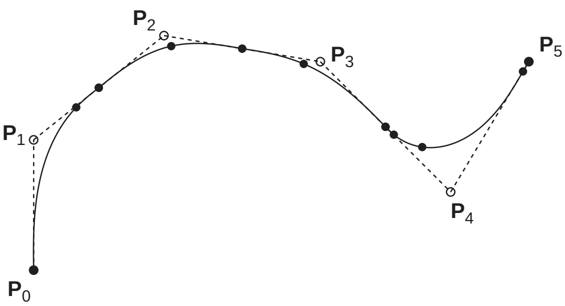

Fig. 9. Example 3—Circular arc inserted at vertex $\mathbf{P}_4$.

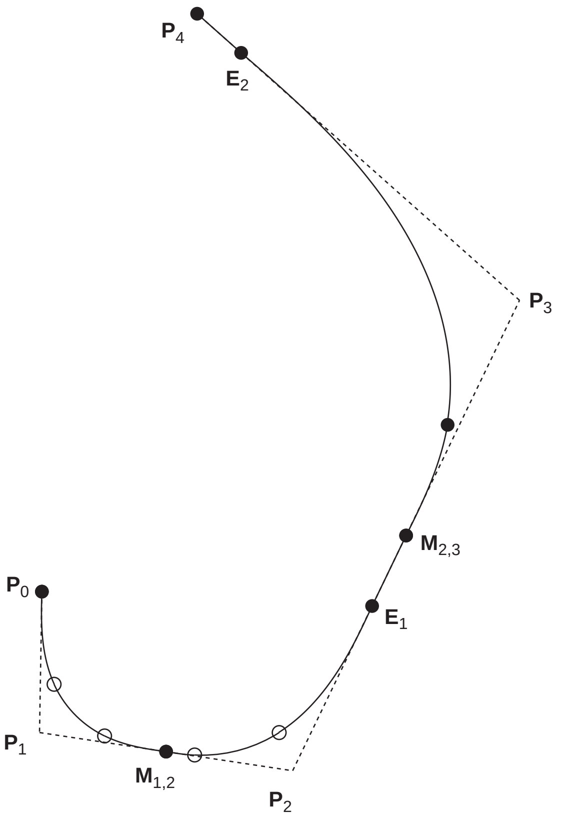

Fig. 10. Example 4—Highway design example.

between the two circles as described in [9]. Also, in the original example, corresponding to vertex $\mathbf{P}_3$, symmetric blending using two clothoids, forms the transition curve from a point on edge $\mathbf{P}_2\mathbf{P}_3$ to a point on edge $\mathbf{P}_3\mathbf{P}_4$; the radius of curvature at the joint is 68.29 units. Some of the shortcomings of this approach are (a) it is not always possible to find a pair of clothoids to form the transition curve between two circles, and (b) it was necessary to specify an additional control point on $\mathbf{P}_2\mathbf{P}_3$ to facilitate symmetric blending.

Using the proposed approach with $\tau = 0.5$, initial unsymmetric blending was done from $\mathbf{P}_1$ to $\mathbf{M}_{1,2}$, the midpoint of $\mathbf{P}_1\mathbf{P}_2$, from $\mathbf{M}_{1,2}$ to $\mathbf{M}_{2,3}$, the midpoint of $\mathbf{P}_2\mathbf{P}_3$, and from $\mathbf{M}_{2,3}$ to $\mathbf{P}_4$. The radii of curvature at the joints of the clothoid pairs corresponding to vertices $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ were respectively 33.98, 45.81 and 91.36 units. Since the first two radii of curvature were smaller than in the original example, circular arcs were inserted. The radii changed to 44.47 and 60.22 units respectively. The endpoints of the inserted circular arcs are indicated by small circles in Fig. 10. The straight line segments $\mathbf{E}_1\mathbf{M}_{2,3}$ and $\mathbf{E}_2\mathbf{P}_4$ were automatically generated as described in the section on unsymmetric blending; so it was not necessary to specify an additional control point along $\mathbf{P}_2\mathbf{P}_3$ as was the case in the original example.

Example 5. This example demonstrates the use of a control polyline for obstacle avoidance when designing a robot path constructed from clothoids. Fig. 11 shows

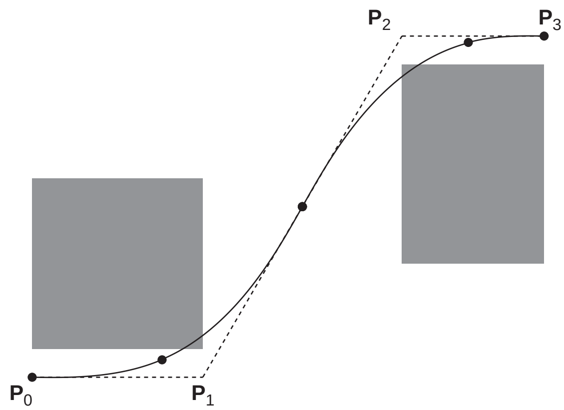

Fig. 11. Example 5—Path with obstacles.

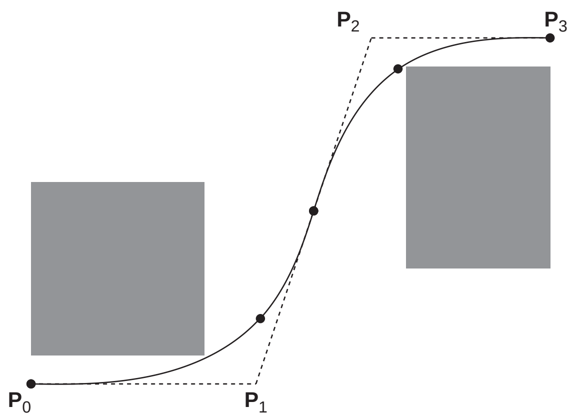

Fig. 12. Example 5—Path avoiding obstacles.

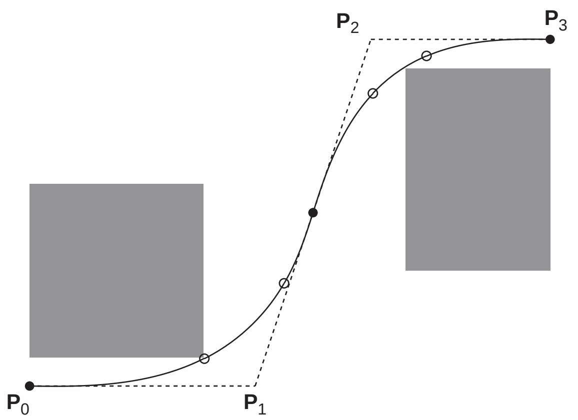

Fig. 13. Example 5—Circular arc insertion in path avoiding obstacles.

two obstacles as grey-shaded rectangles, an initial control polyline as a dashed line, and the path constructed from clothoids. The endpoints of the clothoids are shown as dots. The radii at the joints corresponding to the control polyline vertices $\mathbf{P}_1$ and $\mathbf{P}_2$ are 1.45 and 1.33 units, respectively. It is clear that the path does not avoid the obstacles.

Fig. 12 shows how the control polyline can be modified by repositioning $\mathbf{P}_1$ and $\mathbf{P}_2$ to obtain a path which avoids the obstacles. The radii at the joints corresponding to these vertices are now 1.27 and 1.13, respectively. It can be noticed that these local minimum radii have decreased. In case they are too small, circular arcs can be inserted as described in Section 5 and illustrated in Fig. 13. The endpoints of the circular arcs are shown as small circles. The radii corresponding to the vertices $\mathbf{P}_1$ and $\mathbf{P}_2$ are now 1.65 and 1.36 units, respectively.

7. Conclusion

A control polyline for a clothoid spline has been demonstrated. The advantage of such an approach is that it facilitates interactive design and modification of the spline using a graphics workstation. The modification is local in the sense that the whole curve does not change when a control point is moved.

A new result on clothoid blending using two clothoids which are not the same has also been presented. This result increases the flexibility of using clothoids, the main advantage being fewer straight lines to clothoid transitions.

A further new result was given for inserting a circular arc between two clothoids which are not the same. This is useful for locally adjusting the turning radius in only one place and it affects only those two clothoids.

The resulting curve is composed of clothoids, circular arcs and straight line segments. The offset of such a curve is easily computed and displayed [14].

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References

[1] Scheuer A, Fraichard Th. Continuous-curvature path planning for car-like vehicles. Proceedings of the 1997 IEEE-RSJ conference on intelligent robotics and systems, vol. 2. Grenoble, France; September 1997. p. 997–1003.
[2] Fleury S, Souères P, Laumond J-P, Chatila R. Primitives for smoothing mobile robot trajectories. IEEE Transactions on Robotics and Automation 1995; 11(3):441–8.
[3] Degtiariova-Kostova E, Kostov V. Irregularity of optimal trajectories in a control problem for a car-like robot. INRIA Research Report No. 3411; Sophia Antipolis, France, April 1998.
[4] Riesenfeld R. Applications of B-spline approximation to geometric problems of computer aided design. PhD thesis, Syracuse University, 1973.
[5] Farin G. Curves and surfaces for computer aided geometric design: a practical guide, fourth ed. San Diego: Academic Press; 1997.
[6] Walton DJ, Meek DS. Computer-aided design for horizontal alignment. ASCE Journal of Transportation Engineering 1989;115(4):411–24.
[7] Walton DJ, Meek DS. Clothoidal splines. Computers & Graphics 1990;14(1):95–100.
[8] Meek DS, Thomas RSD. A guided clothoid spline. Computer Aided Geometric Design 1991;8: 163–74.
[9] Meek DS, Walton DJ. The use of Cornu spirals in drawing planar curves of controlled curvature. Journal of Computational and Applied Mathematics 1989;25: 69–78.
[10] Abramowitz M, Stegun IA. Handbook of mathematical functions. New York: Dover; 1965.
[11] Meek DS, Walton DJ. Clothoid spline transition spirals. Mathematics of Computation 1992;59:117–33.
[12] Hickerson TF. Route location and design. New York: McGraw-Hill; 1964.
[13] Papadopoulos E, Poulakakis I. Planning and obstacle avoidance for mobile robots. Proceedings of the IEEE international conference on robotics and automation, ICRA 01. Seoul, Korea, May 2001
[14] Meek DS, Walton DJ. Offset curves of clothoidal splines. Computer-Aided Design 1990;22(4):199–201.

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